Simulation and fixpoint semantics

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Chapter 2

Relators and extensions

In this chapter we introduce the notion of *monotonic relators*, which plays an important role on the semantical, as well as on the syntactical side. For a given functor, a relator is a function from relations to relations such that the relator-image is a relation over the functor-images of the domain and codomain of the argument (definition 2.1.1.1). Monotonicity of a relator is defined by four properties (definition 2.1.1.2), and is the only condition on relators that we need for developing the theory up to chapter 6 (the four properties are actually proof-generated). For defining a monotonic relator, one can give an explicit definition of a relator and then check the monotonicity conditions. *Extensions* extend the given functor in a certain way, and provide an easier way of defining monotonic relators. This chapter is merely a technical preparation for the theory of the following chapters.

The notion of a monotonic relator is closely related to ordered bisimulation as introduced in [Rut94], but I cannot define it in terms of ordered bisimulations. One reason for this is that ordered bisimulations are defined in the category of complete ordered sets with continuous functions, while we consider the category of (pre)ordered sets with monotonic functions. One specific relator, the *minimal* relator (definition 2.1.2), corresponds exactly to bisimulation as given in [AM89]. Bisimulation is introduced by Park in [Par81] . It describes an equivalence on coalgebras, and has the property that two equivalent coalgebras have equal final semantics. The relators in this thesis have a similar use: in terms of relators we define a preorder on coalgebras, and then any semantics (including the final) respects this preorder.

In section 2.1 we introduce monotonic relators in an axiomatic way, and from these axioms we prove the essential properties of relators. The results of this section are all we need for proving properties of a monotonic relator. In section 2.2 we give a way to construct relators by means of extensions. We prove that this construction of relators is complete in the sense that for any monotonic relator there exists a monotonic extension such that the relator can be constructed from this extension. Finally, in section 2.3 we give some relators for the functors $\mathcal{O}$, $\mathcal{P}$, and $\mathcal{Q}$.

The name "relator" is also used by Backhouse [BW92], and also represents a function on (generalized) relations. Despite of some similarities between our "monotonic relators" and Backhouse his "relators", there seems to be no connection.

Let in this chapter an endofunctor F on $\mathbf{Set}$ be given.

2.1 Monotonic relators

An F-relator assigns to a relation between two sets, a relation between the images under F of the two sets. We give the formal definition of a relator, state the four conditions that together define monotonicity of a relator, and define transposition of relators.

Definition 2.1.1

1. *An F-relator Γ is a function that assigns relations to relations, such that for every relation $R \subseteq A \times B$ we have $\Gamma.R \subseteq (F.A) \times (F.B)$.*
2. *A relator Γ is monotonic iff the following properties hold.*
   (a) *If $R \subseteq S$ then $\Gamma.R \subseteq \Gamma.S$ for $R, S \subseteq A \times B$.*
   (b) *$1_{(F.A)} \subseteq \Gamma.(1_A)$ for a set A .*
   (c) *$\Gamma.(R \circ S) \equiv (\Gamma.R) \circ (\Gamma.S)$ for $R \subseteq A \times B$ and $S \subseteq B \times C$.*
   (d) *Let $R \subseteq A \times B$, $f : A \to A'$, and $g : B \to B'$. Then*
   $$((F.f) \times (F.g))[\Gamma.R] \quad \subseteq \quad \Gamma.((f \times g)[R]) \ .$$
3. *For a relator Γ define $\Gamma^{\sim}$ as $R \mapsto (\Gamma.(R^{\sim}))^{\sim}$.*

We get an alternative definition of monotonicity of a relator by replacing the set equality in monotonicity condition (c) by the set containment "$\supseteq$". With this alternative definition we get a larger class of monotonic relators, and this class contains relators that cannot be described by extensions (which is not the case with our definition). If we leave out all the results about extensions (including theorem 2.1.4), all the other results up to chapter 5 still hold. Chapter 5 contains the important soundness result, and this result consist of two parts: soundness for flat models and soundness for complete models. With the alternative definition of monotonicity, soundness for flat models cannot be proved anymore, while soundness for complete models still holds. Because of the importance of soundness, we have chosen the above definition.

The fourth monotonicity condition can probably be formulated more compactly in terms of category theory, but we prefer an explicit definition only using basic category theory. Note that this condition can also be written as: for every $u \in F.A$ and $v \in F.B$ we have

$$u(\Gamma.R)v \quad \Rightarrow \quad (F.f.u)(\Gamma.((f \times g)[R]))(F.g.v) \ .$$

For a relator Γ and a relation $R \subseteq A \times B$, we have $R^{\sim} \subseteq B \times A$, and hence we get $\Gamma.(R^{\sim}) \subseteq F.B \times F.A$, and thus $(\Gamma.(R^{\sim}))^{\sim} \subseteq F.A \times F.B$. So the transposed relator is indeed again a relator.

From the above definition and some calculus it follows that the composition of two relators is again a relator. We prove that monotonicity of relators is preserved by transposition and composition of relators.

Theorem 2.1.1 *Let Γ be a monotonic F-relator.*

1. *The F-relator $\Gamma^{\sim}$ is monotonic.*

2. *Let F' be another endofunctor on* **Set** *and Γ' a monotonic F'-relator. Then $\Gamma \circ \Gamma'$ is a monotonic $F \circ F'$-relator.*

Proof. Under the premise that the properties of definition 2.1.1.2 hold for Γ and Γ', we have to prove that these properties hold for $\Gamma^{\cup}$ and $\Gamma \circ \Gamma'$. This is proved straightforwardly, and we leave them to the reader, except for the fourth property. Let $R \subseteq A \times B$, $f : A \to A'$, and $g : B \to B'$. Then

1. With some calculus we see that, for functions h, j and relation S, we have $(h \times j)[R^{\cup}] \equiv ((j \times h)[R])^{\cup}$. From this fact, some calculus, and the monotonicity of Γ we get the following. $((F.f) \times (F.g))[(\Gamma.(R^{\cup}))^{\cup}] \equiv (((F.g) \times (F.f))[\Gamma.(R^{\cup})])^{\cup} \subseteq (\Gamma.((g \times f)[R^{\cup}]))^{\cup} \equiv (\Gamma.((f \times g)[R])^{\cup}))^{\cup}$.
2.

$$
\begin{array}{ll}
 & (((F \circ F').f) \times ((F \circ F').g))[(\Gamma \circ \Gamma').R] \\
\subseteq & \quad \{ \text{ definition 2.1.1.2.d with, } f := F'.f,\ g := F'.g, \text{ and } R := \Gamma'.R \ \} \\
 & \Gamma.(((F'.f) \times (F'.g))[\Gamma'.R]) \\
\subseteq & \quad \{ \text{ parts a and d of definition 2.1.1.2 } \} \\
 & (\Gamma \circ \Gamma').((f \times g)[R])
\end{array}
$$

□

For $R \subseteq A \times B$ we have $\pi_1 : R \to A$ and $\pi_2 : R \to B$, and thus $\langle F.\pi_1, F.\pi_2 \rangle : F.R \to F.A \times F.B$. From this we see that each endofunctor F induces a specific F-relator, called the minimal relator. Theorem 2.1.2 below justifies this name.

Definition 2.1.2 *Define the minimal F-relator Γ_m by the following. For a relation R we have*

$$\Gamma_m.R \equiv \langle F.\pi_1, F.\pi_2 \rangle [F.R] \ .$$

We prove that the minimal relator is contained in every monotonic relator.

Theorem 2.1.2 *Let Γ be a monotonic F-relator. Then, for a relation R, we have $\Gamma_m.R \subseteq \Gamma.R$.*

Proof. Let $R \subseteq A \times B$. From definition 2.1.1.2.d with $R := 1_R$, $f := \pi_1$, and $g := \pi_2$ we get

$$((F.\pi_1) \times (F.\pi_2))[\Gamma.(1_R)] \subseteq \Gamma.((\pi_1 \times \pi_2)[1_R]) \ .$$

From $(\pi_1 \times \pi_2)[1_R] \equiv R$ and definition 2.1.1.2.b we see that, for $x \in F.R$, we have $(F.\pi_1.x)(\Gamma.R)(F.\pi_2.x)$. From definition 2.1.2 we see that this is exactly what we have to prove. □

Relators, in particular minimal relators, are in general not monotonic, and in the next section we give a necessary and sufficient condition for monotonicity of relators. In the case of the minimal F-relator, this condition exactly states that F preserves pullbacks (a term from category theory [ML71]). However, except for the "$\supseteq$-part" of definition 2.1.1.2.c, all the properties of monotonicity are always satisfied by the minimal relator.

Theorem 2.1.3 *The minimal F-relator satisfies the properties in definition 2.1.1.2, except the "$\supseteq$-part" of property (c).*

Proof.

a Follows from definition 2.1.2 and the general assumption (see section 1.4) that, if $A \subseteq B$ then $F.A \subseteq F.B$.

b Follows from lemma 2.1.1.1 below.

c (the "$\subseteq$-part") Let $u \in F.A$ and $w \in F.C$ be such that $u(\Gamma_m.(R \circ S))w$. We have to prove $u((\Gamma_m.R) \circ (\Gamma_m.S))w$. From definition 2.1.2 we get $z \in F.(R \circ S)$ such that $F.\pi_1.z \equiv u$ and $F.\pi_2.z \equiv w$. Observe that in general there exists a function $f : R \circ S \to B$ such that $\forall (a,c) \in R \circ S\, (aR(f.(a,c)) \wedge (f.(a,c))Sc)$. Equivalently: f satisfies the two properties $\langle \pi_1, f\rangle : R \circ S \to R$ and $\langle f, \pi_2\rangle : R \circ S \to S$. It suffices to prove (1) $u(\Gamma_m.R)(F.f.z)$ and (2) $(F.f.z)(\Gamma_m.S)w$. From symmetry we see that it suffices to prove one of them, say (1) . From definition 2.1.2 we see that it suffices to find $x \in F.R$ such that (a) $F.\pi_1.x \equiv u$ and (b) $F.\pi_2.x \equiv F.f.z$. Put $x \equiv F.\langle \pi_1, f\rangle.z$. With some calculus we see that $F.\pi_1.x \equiv F.\pi_1.z$ and $F.\pi_2.x \equiv F.f.z$. Finally, note that $F.\pi_1.z \equiv u$.

d Let $R \subseteq A \times B$, $f : A \to A'$, and $g : B \to B'$. Assume that $u \in F.A$ and $v \in F.B$ are such that $u(\Gamma_m.R)v$. We have to prove $(F.f.u)(\Gamma_m.((f \times g)[R]))(F.g.v)$. From $u(\Gamma_m.R)v$ and definition 2.1.2 we get $x \in F.R$ such that $F.\pi_1.x \equiv u$ and $F.\pi_2.x \equiv v$. Put $y \equiv F.(f \times g).x$ (so $y \in F.((f \times g)[R])$) . From definition 2.1.2 we see that it suffices to prove $F.\pi_1.y \equiv F.f.u$ and $F.\pi_2.y \equiv F.g.v$. With some calculus we see that $F.\pi_1.y \equiv F.\pi_1.(F.(f \times g).x) \equiv F.(f \circ \pi_1).x \equiv F.f.u$. The proof of $F.\pi_2.y \equiv F.g.v$ is symmetric.

□

We prove that the minimal relator respects the identity relation and transposition of relations. These properties do not hold for general relators. Actually, in the next section we see that the first property uniquely determines the minimal relator.

Lemma 2.1.1

1. *For a set A we have $\Gamma_m.(1_A) \equiv 1_{(F.A)}$.*
2. *For a relation R we have $\Gamma_m.(R^{\sim}) \equiv (\Gamma_m.R)^{\sim}$.*

Proof.

1. $\subseteq$ **:** From definition 2.1.2 we see that it suffices to prove that, for $x \in F.(1_A)$, we have $F.\pi_1.x \equiv F.\pi_2.x$. Because π_1 and π_2 are equal as functions from 1_A to A, the functions $F.\pi_1$ and $F.\pi_2$ are equal as functions from $F.(1_A)$ to $F.A$.

   $\supseteq$ **:** From definition 2.1.2 we see that we have to prove that, for $u \in F.A$, there exists $x \in F.1_A$ such that $F.\pi_1.x \equiv u$ and $F.\pi_2.x \equiv u$. Let $u \in F.A$ and define $x \equiv F.\langle id_A, id_A\rangle.u$. Then $F.\pi_1.x \equiv F.\pi_1.(F.\langle id_A, id_A\rangle.u) \equiv$ $F.(\pi_1 \circ \langle id_A, id_A\rangle).u \equiv F.id_A.u \equiv u$. From symmetry the equality $F.\pi_2.x \equiv u$ follows.

2. Let $R \subseteq A \times B$. It suffices to prove one containment, say $(\Gamma_m.R)^\sim \subseteq \Gamma.(R^\sim)$, because then by substituting $R^\sim$ for R and transposing the containment, the other containment follows. Let $x \in F.R$. From definition 2.1.2 we see that it suffices to find $x' \in F.(R^\sim)$ such that $F.\pi_1.x' \equiv F.\pi_2.x$ and $F.\pi_2.x' \equiv F.\pi_1.x$. Note that $\langle \pi_2, \pi_1 \rangle : R \to R^\sim$, and define $x' \equiv F.\langle \pi_2, \pi_1 \rangle.x$. Then $F.\pi_1.x' \equiv F.\pi_1.(F.\langle \pi_2, \pi_1 \rangle.x) \equiv F.(\pi_1 \circ \langle \pi_2, \pi_1 \rangle).x \equiv F.\pi_2.x$. The other equality follows from symmetry.

$\square$

From part 2 of the above theorem it follows that the minimal relator maps equivalence relations to equivalence relations. We mentioned that the above properties do not hold for general relators, and this is also the case for this deduced property (see example 2.3.1).

For each set A, a relator induces a relation on $F.A$ by applying the relator to the identity relation 1_A . If this relator is monotonic then we can prove that this relation is a preorder, but because we do not yet need this result, the proof is postponed to the next section. However, the notation already reflects this result.

Definition 2.1.3 *For a set A and an F-relator Γ define the relation $\lhd_{\Gamma,A} \equiv \Gamma.(1_A)$ on $F.A$.*

We prove that the above defined family of relations is type-insensitive, and consequently that we can simplify its notation. With type-insensitive we mean that the relation $\lhd_{\Gamma,A}$ does not depend on the set A . This is formalized and proved in the following lemma.

Lemma 2.1.2 *Let Γ be a monotonic relator, A, B sets such that $A \subseteq B$, and let $u, u' \in F.A$. Then*

$$u \lhd_{\Gamma,A} u' \quad \Leftrightarrow \quad u \lhd_{\Gamma,B} u' .$$

Proof. Note that $F.A \subseteq F.B$, and so $u \lhd_{\Gamma,B} u'$ makes sense.

$\Rightarrow$: Because $(id_A \times id_A)[1_A] \subseteq 1_B$ and $F.id_A \equiv id_{(F.A)}$, from definition 2.1.1.2.d we see that $u(\Gamma.(1_A))u'$ implies $u(\Gamma.(1_B))u'$. Finally, use definition 2.1.3 .

$\Leftarrow$: Note that from definition 2.1.3 and definition 2.1.1.2.b it follows that $\lhd_{\Gamma,A}$ is reflexive. Because $1_A \circ 1_B \equiv 1_A$, from definition 2.1.1.2.c and definition 2.1.3 we see that $\lhd_{\Gamma,A} \circ \lhd_{\Gamma,B} \equiv \lhd_{\Gamma,A}$. Finally, use the fact that $\lhd_{\Gamma,A}$ is reflexive.

$\square$

Let u and u' be elements of $F.A$ and also elements of $F.B$. Then u and u' are also elements of $F.(A \cup B)$, and from the above lemma we see that $u \lhd_{\Gamma,A} u' \Leftrightarrow u \lhd_{\Gamma,A\cup B} u' \Leftrightarrow u \lhd_{\Gamma,B} u'$. So we can omit the sub-index "A" in the notation "$\lhd_{\Gamma,A}$", and we write "$\lhd_\Gamma$" . Furthermore, if for a set A, we apply a monotonic relator Γ to the relation 1_A then we also omit the sub-index "A" in "1_A", and write $\Gamma.1$. So definition 2.1.3 becomes $\lhd_\Gamma \equiv \Gamma.1$.

We prove that any monotonic relator Γ can be expressed in terms of the minimal relator and the family of preorders $\lhd_\Gamma$. In section 2.2 we prove the reverse result, that is, we prove that any monotonic relator can be constructed in this way.

Theorem 2.1.4 *Let Γ be a monotonic F-relator. Then, for a relation R, we have*

$$\Gamma.R \quad \equiv \quad \lhd_\Gamma \circ (\Gamma_m.R) \circ \lhd_\Gamma .$$

Proof. In the proof we omit the sub-index "Γ" in "$\lhd_\Gamma$" . Let $R \subseteq A \times B$.

$\Rightarrow$: Let $u_0 \in F.A$ and $v_0 \in F.B$ be such that $u_0(\Gamma.R)v_0$. From definition 2.1.2 we see that we have to find $x_0 \in F.R$ such that $u_0 \lhd F.\pi_1.x_0$ and $F.\pi_2.x_0 \lhd v_0$. Define $S \subseteq A \times R$ and $T \subseteq R \times B$ as follows.

- $S \equiv \{(a,(a,b)) \mid (a,b) \in R\}$.
- $T \equiv \{((a,b),b) \mid (a,b) \in R\}$.

We see that $S \circ T \equiv R$ holds, and from definition 2.1.1.2.c we get $(\Gamma.S) \circ (\Gamma.T) \equiv \Gamma.R$. Because $u_0(\Gamma.R)v_0$ holds, we get $x_0 \in F.R$ such that $u_0(\Gamma.S)x_0$ and $x_0(\Gamma.T)v_0$. From definition 2.1.1.2.d with $R := S$, $f := id_A$, and $g := \pi_1$, we get

$$((F.id_A) \times (F.\pi_1))[\Gamma.S] \subseteq \Gamma.((id_A \times \pi_1)[S]) \qquad (*) \ .$$

From the definition of S we see that $(id_A \times \pi_1)[S] \subseteq 1_A$. So from $(*)$ we see that, for $u \in F.A$ and $x \in F.R$ such that $u(\Gamma.S)x$, we have $u \lhd F.\pi_1.x$. Similarly we can deduce that, for $x \in F.R$ and $v \in F.B$ such that $x(\Gamma.T)v$, we have $F.\pi_2.x \lhd v$. We already had the existence of $x_0 \in F.R$ such that $u_0(\Gamma.S)x_0$ and $x_0(\Gamma.T)v_0$, which implies $u_0 \lhd F.\pi_1 x_0$ and $F.\pi_2.x_0 \lhd v_0$.

$\Leftarrow$: We have to prove that $\lhd \circ (\Gamma_m.R) \circ \lhd \subseteq \Gamma.R$. This follows directly from theorem 2.1.2 and lemma 2.1.3 below.

□

One half of the above theorem is proved in the following lemma.

Lemma 2.1.3 *Let Γ be a monotonic F-relator. Then* $\lhd_\Gamma \circ (\Gamma.R) \equiv (\Gamma.R) \circ \lhd_\Gamma \equiv \Gamma.R$.

Proof. From definition 2.1.3, definition 2.1.1.2.c, and some calculus we get $\lhd_\Gamma \circ (\Gamma.R) \equiv (\Gamma.1) \circ (\Gamma.R) \equiv \Gamma.(1 \circ R) \equiv \Gamma.R$. The other conjunct is proved similar. □

The following lemma is not needed yet, but it states a very important property of monotonic relators: monotonicity on functions.

Lemma 2.1.4 *Let $R \subseteq A \times B$, $f : X \to A$, and $g : X \to B$. Then*

$$fRg \quad \Rightarrow \quad (F.f)(\Gamma.R)(F.g) \ .$$

Proof. Note that fRg can be written as $(f \times g)[1_X] \subseteq R$, and $(F.f)(\Gamma.R)(F.g)$ as $((F.f) \times (F.g))[1_{(F.X)}] \subseteq \Gamma.R$. Then

$$\begin{array}{ll} & ((F.f) \times (F.g))[1_{(F.X)}] \subseteq \Gamma.R \\ \Leftarrow & \quad \{ \text{ definition 2.1.1.2.b, calculus } \} \\ & ((F.f) \times (F.g))[\Gamma.(1_X)] \subseteq \Gamma.R \\ \Leftarrow & \quad \{ \text{ parts d and a of definition 2.1.1.2 } \} \\ & (f \times g)[1_X] \subseteq R \end{array}$$

□

2.2 Extensions

In the previous section we proved that a monotonic relator Γ induces a family of relations $\lhd_\Gamma$ (definition 2.1.3). In this section we start with a family of relations that satisfy a certain property. The family of relations is given by an *extension*, and the property is called *monotonicity* of the extension. In theorem 2.1.4 we proved that a monotonic relator can be expressed in terms of the induced family of relations and the minimal relator. In this section we use the equivalence of that theorem for constructing a monotonic relator from a monotonic extension. Furthermore, we prove that any monotonic relator can be constructed from a monotonic extension.

An extension of the endofunctor F adds a family of preorders to the definition of F , where this family of preorders is type-insensitive (see lemma 2.1.2).

Definition 2.2.1 *Let* G *be a functor from* **Set** *to* **Prs**, *say* $G.A \equiv (\_, \lhd_{G,A})$. *We define* G *to be an extension of* F *iff the following holds.*

- $G.A \equiv (F.A, \lhd_{G,A})$ *for a set* A
- *For sets* A, B *with* $A \subseteq B$ *and* $u, u' \in F.A$ *we have*

$$u \lhd_{G,A} u' \quad \Leftrightarrow \quad u \lhd_{G,B} u' \; .$$

- $G.f \equiv F.f$ *for a function* f .

In the previous section, from lemma 2.1.2 we concluded that the sub-index "A" could be omitted in "$\lhd_{\Gamma,A}$" . In the same way, from the above property in definition 2.2.1, we conclude that the sub-index "A" can be omitted in "$\lhd_{G,A}$", and we write "$\lhd_G$" . It is easily proved that the functor $G : \mathbf{Set} \to \mathbf{Prs}$ defined by $\lhd_{G,A} \equiv 1_{(F.A)}$, is an extension. We denote this extension also by F, and call it the trivial extension of F .

In theorem 2.1.4 a monotonic relator Γ is expressed in terms of the minimal relator and the family of relations $\lhd_\Gamma$. Given an extension of F, with the equivalence of that theorem we now define a relator.

Definition 2.2.2 *Let* G *be an extension of* F . *Define the* F*-relator* $R \mapsto R^G$ *by the following. For a relation* R *we have*

$$R^G \quad \equiv \quad \lhd_G \circ (\Gamma_m.R) \circ \lhd_G \; .$$

For an extension G we see that from lemma 2.1.1.1 and the fact that $\lhd_G$ is a preorder, it follows that $\lhd_G \equiv 1^G$ (note the resemblance with definition 2.1.3). In this section we assume this fact to be known. If in the above definition we substitute the trivial extension F for G (so $\lhd_G := 1$) then we see that the relator $R \mapsto R^F$ is the minimal relator Γ_m .

In theorem 2.1.3 we proved that the minimal relator is, except for one property, monotonic. We prove that the above defined relator, which is based on the minimal relator, inherits this property.

Theorem 2.2.1 *Let* G *be an extension of* F . *The* F*-relator* $R \mapsto R^G$ *satisfies the properties in definition 2.1.1.2, except for the "*$\supseteq$*-part" of property (c) .*

Proof. The properties (a) and (b) in definition 2.1.1 follow easily from definition 2.2.2 and theorem 2.1.3, and we only prove (1) property (c) (the "$\subseteq$-part") and (2) property (d) . In the proof we omit the sub-index "G" in "$\lhd_G$" .

1. Let $R \subseteq A \times B$ and $S \subseteq B \times C$. Then

$$\begin{array}{ll} & (R \circ S)^G \\ \subseteq & \quad \{ \text{ definition 2.2.2, theorem 2.1.3, calculus } \} \\ & \lhd \circ (\Gamma_m.R) \circ (\Gamma_m.S) \circ \lhd \\ \subseteq & \quad \{ \ 1_{(F.B)} \subseteq \lhd \circ \lhd, \text{ calculus } \} \\ & \lhd \circ (\Gamma_m.R) \circ \lhd \circ \lhd \circ (\Gamma_m.S) \circ \lhd \\ \equiv & \quad \{ \text{ two times definition 2.2.2 } \} \\ & R^G \circ S^G \end{array}$$

2. Let $R \subseteq A \times B$, $f : A \to A'$, and $g : B \to B'$. Assume that $u \in F.A$ and $v \in F.B$ are such that $uR^G v$. Put $S \equiv (f \times g)[R]$, so we have to prove $(F.f.u)S^G(F.g.v)$. From $uR^G v$ and definition 2.2.2 we get $u' \in F.A$ and $v' \in F.B$ such that (a) $u \lhd u'$, (b) $u'(\Gamma_m.R)v'$, and (c) $v' \lhd v$. Because G is a functor, from (a) and (c) we get (a′) $F.f.u \lhd F.f.u'$ and (c′) $F.g.v' \lhd F.g.v$. From theorem 2.1.3 and (b) we get (b′) $(F.f.u')(\Gamma_m.S)(F.g.v')$. Then (a′), (b′), and (c′) together imply $u(\lhd \circ (\Gamma_m.S) \circ \lhd)v$. From definition 2.2.2 we see that this is equivalent to $uS^G v$.

□

For proving monotonicity of the relator of definition 2.2.2, that is, for proving the one missing property, we need an extra condition on the extension, called monotonicity.

Definition 2.2.3 *An extension G of F is defined to be monotonic if the following holds. Let $u \in F.A$, $v \in F.B$, $f : A \to C$, and $g : B \to C$. Then*

$$F.f.u \lhd_G F.g.v \quad \Rightarrow \quad u\{(a,b) \in A \times B \mid f.a \equiv g.b\}^G v \ .$$

From definition 2.2.2, definition 2.1.2, and the above definition, we see that the trivial extension F is monotonic iff for every f, g, u, v we have

$$F.f.u \equiv F.g.v \quad \Rightarrow \quad \exists w \in F.(\{(a,b) \mid f.a \equiv g.b\})\,(F.\pi_1.w \equiv u \wedge F.\pi_2.w \equiv v) \ .$$

In terms of category theory, this property of the functor F is formulated as: F preserves pullbacks (see [ML71]).

With the extra condition of monotonicity of the extension, we can now prove the missing property, that is, monotonicity of the induced relator.

Theorem 2.2.2 *Let G be a monotonic extension of F . The F-relator $R \mapsto R^G$ is monotonic.*

Proof. In the proof we omit the sub-index "G" in "$\lhd_G$" . From theorem 2.2.1 we see that we only have to prove the "$\supseteq$-part" of definition 2.1.1.2.c . Let $R \subseteq A \times B$, $S \subseteq B \times C$, $u \in F.A$, $v \in F.B$, and $w \in F.C$ be such that $uR^G v$ and $vS^G w$. We have to prove $u(R \circ S)^G w$.

From definition 2.2.2 and definition 2.1.2 we see that from uR^Gv we get $x \in F.R$ such that $u \lhd F.\pi_1.x$ and $F.\pi_2.x \lhd v$. In the same way, from vS^Gw we get $y \in F.S$ such that $v \lhd F.\pi_1.y$ and $F.\pi_2.y \lhd w$. Because $\lhd$ is a preorder, we get $F.\pi_2.x \lhd F.\pi_1.y$. Put $T \equiv \{(r,s) \in R \times S \mid \pi_2.r \equiv \pi_1.s\}$. Then from definition 2.2.3 we see that xT^Gy holds. From theorem 2.2.1 we see that the property in definition 2.1.1.2.d can be applied, and we do so with $f := \pi_1$, $g := \pi_2$, and $R := T$. Then we get

$$((F.\pi_1) \times (F.\pi_2))[T^G] \quad \subseteq \quad ((\pi_1 \times \pi_2)[T])^G \ .$$

Because $(\pi_1 \times \pi_2)[T] \equiv R \circ S$, and xT^Gy holds, we see that $(F.\pi_1.x)(R \circ S)^G(F.\pi_2.y)$. Together with definition 2.2.2, $u \lhd F.\pi_1.x$, and $F.\pi_2.y \lhd w$ we see that $u(R \circ S)^G w$. □

In the above theorem we see that from every monotonic extension we can construct a monotonic relator. Because $R \mapsto R^F$ is the minimal relator, we see that monotonicity of the trivial extension F (preserving pullbacks) is a sufficient condition for the minimal relator to be monotonic.

In the following theorem we prove that every monotonic relator can be constructed from a unique monotonic extension.

Theorem 2.2.3 *Let Γ be a monotonic F-relator. There exists a unique extension G of F such that Γ equals $R \mapsto R^G$. Furthermore, this unique extension G is monotonic and defined by $\lhd_G \equiv \lhd_\Gamma$.*

Proof. Let G, G' be two extensions of F such that $(R \mapsto R^G) \equiv \Gamma \equiv (R \mapsto R^{G'})$. Note that if $\lhd_G \equiv \lhd_{G'}$ then G and G' are equal, and observe $\lhd_G \equiv 1^G \equiv 1^{G'} \equiv \lhd_{G'}$. So it suffices to prove existence. From definition 2.2.1, lemma 2.1.2, and theorem 2.2.4 below we see that an extension G is defined by $\lhd_G \equiv \lhd_\Gamma$. From theorem 2.1.4 and definition 2.2.2 we see that the relators $R \mapsto R^G$ and Γ are equal, so it remains to prove that G is monotonic. Let $u \in F.A$, $v \in F.B$, $f : A \to C$, and $g : B \to C$. From definition 2.2.3 we see that we have to prove the following.

$$F.f.u \lhd_\Gamma F.g.v \quad \Rightarrow \quad u(\Gamma.(\{(a,b) \in A \times B \mid f.a \equiv g.b\}))v \ .$$

Define $R \subseteq A \times C$ and $S \subseteq C \times B$ as

- $R \equiv \{(a, f.a) \mid a \in A\}$
- $S \equiv \{(g.b, b) \mid b \in B\}$

From the definition of R we see that $R \equiv (id_A \times f)[1_A]$. From definition 2.1.1.2.d we see that, for $u \in F.D$, we have $u(\Gamma.R)(F.f.u)$. From symmetry we see that, for $v \in F.E$, we have $(F.g.v)(\Gamma.S)v$. Let $u \in F.A$ and $v \in F.B$ be such that $F.f.u \lhd_\Gamma F.g.v$. Because $u(\Gamma.R)(F.f.u)$ and $(F.g.v)(\Gamma.S)v$ hold, we get $u((\Gamma.R) \circ \lhd_\Gamma \circ (\Gamma.S))v$. From lemma 2.1.3 and definition 2.1.1.2.c we get $u(\Gamma.(R \circ S))v$. Finally, from the definitions of R and S it is easily proved that $R \circ S \equiv \{(a,b) \mid f.a \equiv g.b\}$. □

For the proof of following theorem, we do not need any results of this section, only of the previous. We prove that a monotonic relator Γ defines an endofunctor on $\mathbf{Prs}$.

Theorem 2.2.4 *Let Γ be a monotonic F-relator and $(A, \leq_A)$ a preordered set. Then*

1. *$(F.A, \Gamma.(\leq_A))$ is a preordered set.*
2. *Let $(B, \leq_B)$ be a preordered set and $f : A \to B$. If f is monotonic from $(A, \leq_A)$ to $(B, \leq_B)$ then $F.f$ is monotonic from $(F.A, \Gamma.(\leq_A))$ to $(F.B, \Gamma.(\leq_B))$.*

Proof.

1. We have to prove reflexivity: $I_{(F.A)} \subseteq \Gamma.(\leq_A)$, and transitivity: $(\Gamma.(\leq_A)) \circ (\Gamma.(\leq_A)) \subseteq \Gamma.(\leq_A)$. The first statement follows from parts a and b of definition 2.1.1.2, and the second from parts a and c of definition 2.1.1.2 .
2. Note that monotonicity of f is equivalent to $(f \times f)[\leq_A] \subseteq \leq_B$. With definition 2.1.1.2.d this implies that, for $u, u' \in F.A$ such that $u(\Gamma.(\leq_A))u'$, we have $(F.f.u)(\Gamma.(\leq_B))(F.f.u')$. This is exactly what we have to prove.

□

From theorem 2.2.2 we know that each monotonic extension of the endofunctor F on $\mathbf{Set}$, induces a monotonic relator, and with the above theorem we see that this relator induces an endofunctor on $\mathbf{Prs}$. We introduce a notation for this, by a monotonic extension, induced functor.

Definition 2.2.4 *Let G be a monotonic extension of F . Define the endofunctor G^+ on* **Prs** *by the following.*

- $G^+.(A, \leq) \equiv (F.A, \leq^G)$ *for* $(A, \leq) \in obj(\mathbf{Prs})$.
- $G^+.f \equiv F.f$ *for* $f \in arr(\mathbf{Prs})$.

In example 2.3.1.2 below we show that the above defined endofunctor G^+ on the category of preordered sets, is in general not an endofunctor on the category of ordered sets. We actually prove that, for an order $\leq$, the relation $\leq^G$ is in general not an order. This is the main reason for working with preordered sets instead of ordered sets.

For an extension G of F, define the extension $G^\sim$ of F by $\lhd_{(G^\sim)} \equiv (\lhd_G)^\sim$. Note that for the trivial extension F we have $F^\sim \equiv F$. We prove that the transposition of the relator that is induced by a monotonic extension, equals the relator that is induced by the transposed extension. We also state and prove a connection between the composition of two extensions, and the composition of the two induced relators.

Corollary 2.2.1 *Let G be a monotonic extension of F .*

1. *Then the extension $G^\sim$ is monotonic, and the F-relators $R \mapsto R^{(G^\sim)}$ and $(R \mapsto R^G)^\sim$ are equal.*
2. *Let F' be another endofunctor on* **Set** *and G' a monotonic extension of F' . Then $G^+ \circ G'$ is a monotonic extension of $F \circ F'$, and we have*
   (a) $R^{(G^+ \circ G')} \equiv (R^{G'})^G$ *for a relation* R .
   (b) $(G^+ \circ G')^+ \equiv G^+ \circ (G')^+$.

Proof.

1. From theorem 2.1.1 we see that $(R \mapsto R^G)^\sim$ is a monotonic relator. Then from theorem 2.2.3 we know that there exists an extension G' of F such that the relators $R \mapsto R^{G'}$ and $(R \mapsto R^G)^\sim$ are equal. The extension G' is defined by $\lhd_{G'} \equiv ((1^\sim)^G)^\sim) \equiv (\lhd_G)^\sim$. So $G' \equiv G^\sim$.
2. From theorem 2.1.1.2 we see that $R \mapsto (R^{G'})^G$ is a monotonic relator. Then from theorem 2.2.3 we know that there exists a monotonic extension G'' of $F \circ F'$ such that the relators $R \mapsto R^{G''}$ and $R \mapsto (R^{G'})^G$ are equal. It suffices to prove (i) the functors $(G'')^+$ and $G^+ \circ (G')^+$ are equal and (ii) the functors G'' and $G^+ \circ G'$ are equal.

   i : It is easily proved that the two functors are equal on functions, and it remains to prove that the two are equals on preordered sets. Let $(A, \leq)$ be a preordered set. Then

$$\begin{array}{ll} & (G'')^+.(A,\leq) \\ \equiv & \quad \{ \text{ definition 2.2.4 } \} \\ & ((F \circ F').A, \leq^{G''}) \\ \equiv & \quad \{ \leq^{G''} \equiv (\leq^{G'})^G, \text{ definition 2.2.4 } \} \\ & G^+.(F'.A, \leq^{G'}) \\ \equiv & \quad \{ \text{ definition 2.2.4 } \} \\ & (G^+ \circ (G')^+).(A,\leq) \end{array}$$

   ii : It is easily proved that the two functors are equal on functions, and it remains to prove that the two are equal on sets. Let A be a set. If we apply the two equal functors (see (a)) $(G'')^+$ and $G^+ \circ (G')^+$ to the preordered set $(A, 1_A)$ then from definition 2.2.4 we see that we get $G''.A$ and $G^+.(G'.A)$, respectively.

$\square$

Let F' be another endofunctor on $\mathbf{Set}$ (next to F) such that the trivial extension F' is monotonic, and let G be a monotonic extension of F . From the above corollary with $G' := F'$ we see that $G^+ \circ F'$ is again a monotonic extension. For a set A we see that $(G^+ \circ F').A \equiv G^+.(F'.A, 1_{(F'.A)}) \equiv (F.(F'.A), 1^G) \equiv (F.(F'.A), \lhd_G) \equiv G.(F'.A)$. Because the functors $G^+ \circ F'$ and $G \circ F'$ are also equal on arrows, we see that they are equal (F' typed as $\mathbf{Set} \to \mathbf{Prs}$ and $\mathbf{Set} \to \mathbf{Set}$, respectively). Obviously, we prefer the second notation.

2.3 Some extensions of $\mathcal{O}$, $\mathcal{P}$, and $\mathcal{Q}$

In this section we look at some extensions of the endofunctors $\mathcal{O}$, $\mathcal{P}$, and $\mathcal{Q}$, and the corresponding relators. Each endofunctor has a trivial extension, which corresponds to the minimal relator. For the endofunctor $\mathcal{P}$ we define a non-trivial extension, which composed with $\mathcal{O}$ also defines a non-trivial extension of $\mathcal{Q}$. For all these extensions (three trivial and two non-trivial) we give an explicit expression for the corresponding relator, and prove that they are monotonic (both the extension and relator).

We first define the non-trivial extensions of $\mathcal{P}$ and $\mathcal{Q}$. The family of preorders of both extensions is the subset relation. Because the subset relation is obviously type-insensitive, from definition 2.2.1 we see that the two are indeed extensions.

Definition 2.3.1

1. *Define the extension* $\mathcal{P}c$ *of* $\mathcal{P}$ *by* $\lhd_{\mathcal{P}c} \equiv \subseteq$.
2. *Define the extension* $\mathcal{Q}c$ *of* $\mathcal{Q}$ *as* $\mathcal{Q}c \equiv \mathcal{P}c \circ \mathcal{O}$.

In definition 2.2.2 we defined a relator in terms of the minimal relator and a family of preorders (given by an extension). In the following result we have a reversed situation: we express the minimal $\mathcal{P}$-relator in terms of the relator that is induced by the extension $\mathcal{P}c$. This result is used for proving results about $\mathcal{P}$, by reducing the case $\mathcal{P}$ to the case $\mathcal{P}c$, which is often much simpler.

Theorem 2.3.1 *For a relation* R *we have* $R^{\mathcal{P}} \equiv \mathcal{R}^{\mathcal{P}c} \cap R^{(\mathcal{P}c^{\sim})}$.

Proof. We first prove a generalization of one half of the equality. Let F be an endofunctor on **Set**, G an extension of F, and $R \subseteq A \times B$. From definition 2.2.2 we get (1) $R^G \equiv \lhd \circ R^F \circ \lhd$. If we substitute $R := R^{\sim}$ in (1), and use corollary 2.2.1, lemma 2.1.1.2, and some calculus then we get (2) $R^{(G^{\sim})} \equiv \lhd^{\sim} \circ R^F \circ \lhd^{\sim}$. Because $\lhd$ is a preorder, (1) and (2) imply $R^F \subseteq R^G \cap R^{(G^{\sim})}$. Now substitute $F := \mathcal{P}$ and $G := \mathcal{P}c$, so it remains to prove $R^{\mathcal{P}c} \cap R^{(\mathcal{P}c^{\sim})} \subseteq R^{\mathcal{P}}$. Let $u \in \mathcal{P}.A$ and $v \in \mathcal{P}.B$ be such that $u(R^{\mathcal{P}c} \cap R^{(\mathcal{P}c^{\sim})})v$. We have to prove $uR^{\mathcal{P}}v$. From (1), (2), and definition 2.1.2 we get $x, y \in \mathcal{P}.R$ such that

$$u \lhd_{\mathcal{P}c} \mathcal{P}.\pi_1.x \quad \wedge \quad \mathcal{P}.\pi_2.x \lhd_{\mathcal{P}c} v \quad \wedge \quad \mathcal{P}.\pi_1.y \lhd_{\mathcal{P}c} u \quad \wedge \quad v \lhd_{\mathcal{P}c} \mathcal{P}.\pi_2.y \; .$$

With definition 1.4.4.1 and definition 2.3.1.1 this becomes

$$\pi_1[y] \subseteq u \subseteq \pi_1[x] \quad \wedge \quad \pi_2[x] \subseteq v \subseteq \pi_2[y] \; .$$

Now put $z \equiv (x \cup y) \cap (u \times v)$. Because $u \in \mathcal{P}.A$ and $v \in \mathcal{P}.B$ (so $\#u, \#v < \rho$), from $z \subseteq u \times v$ it follows that $\#z < \#(u \times v) < \rho^2 \leq \rho$. Then from $z \subseteq x \cup y \subseteq R$, we see that $z \in \mathcal{P}.R$. With some calculus it is easily proved that $\pi_1[z] \equiv u$ and $\pi_2[z] \equiv v$, which with definition 2.1.2 implies $uR^{\mathcal{P}}v$. □

We give explicit expressions for the relators induced by $\mathcal{O}$, $\mathcal{P}$, and $\mathcal{P}c$.

Theorem 2.3.2 *Let* $R \subseteq A \times B$, $(p, I) \in \mathcal{O}.A$, $(q, J) \in \mathcal{O}.B$, $u \in \mathcal{P}.A$, *and* $v \in \mathcal{P}.B$ *we have*

1. $(p, I)R^{\mathcal{O}}(q, J) \quad \Leftrightarrow \quad (p \equiv q) \wedge IRJ$.
2. $uR^{\mathcal{P}c}v \quad \Leftrightarrow \quad \forall a \in u \exists b \in v \, (aRb)$.
3. $uR^{\mathcal{P}}v \quad \Leftrightarrow \quad \forall a \in u \, \exists b \in v \, (aRb) \wedge \forall b \in v \, \exists a \in u \, (aRb)$.

Proof.

1.

$$
\begin{array}{ll}
 & (p,I)R^{\mathcal{O}}(q,J) \\
\Leftrightarrow & \quad \{ \text{definition 2.2.2, definition 2.1.2} \} \\
 & \exists x \in \mathcal{O}.R\,(\mathcal{O}.\pi_1.x \equiv (p,I) \wedge \mathcal{O}.\pi_2.x \equiv (q,J)) \\
\Leftrightarrow & \quad \{ \text{definition 1.4.3, calculus} \} \\
 & \exists r \in \mathsf{Op}, L \in R^{\eta.r}\,((r,\pi_1 \circ L) \equiv (p,I) \wedge (r,\pi_2 \circ L) \equiv (q,J)) \\
\Leftrightarrow & \quad \{ \text{calculus} \} \\
 & (p \equiv q) \wedge IRJ
\end{array}
$$

2.

$$
\begin{array}{ll}
 & uR^{\mathcal{P}c}v \\
\Leftrightarrow & \quad \{ \text{definition 2.2.2} \} \\
 & \exists x \in \mathcal{P}.R\,(u \lhd_{\mathcal{P}c} \mathcal{P}.\pi_1.x \wedge \mathcal{P}.\pi_2.x \lhd_{\mathcal{P}c} v) \\
\Leftrightarrow & \quad \{ \text{definition 2.3.1} \} \\
 & \exists x \subseteq R\,(\#x < \rho \wedge u \subseteq \pi_1[x] \wedge \pi_2[x] \subseteq v) \\
\Leftrightarrow & \quad \{ \text{“}\Rightarrow\text{:” calculus, “}\Leftarrow\text{:” see below} \} \\
 & \forall a \in u\,\exists b \in v\,(aRb)
\end{array}
$$

It remains to prove the implication in the above proof. Assume $\forall a \in u\,\exists b \in v\,(aRb)$, and put $x \equiv R \cap (u \times v)$. Because $u \in \mathcal{P}.A$ and $v \in \mathcal{P}.B$ (so $\#u, \#v < \rho$), from $x \subseteq u \times v$ we see that $\#x \leq \#(u \times v) < \rho^2 \leq \rho$. From the definition of x we see that $\pi_2[x] \subseteq v$, and the assumption on u and v implies that $u \subseteq \pi_1[x]$.

3. Follows immediately from theorem 2.3.1 and part 2.

□

In lemma 2.1.1.2 we proved that the minimal relator respects the transposition of relations. We show that this property does not hold for general relators. We also give two other cases that might cause confusion when using the relator notation.

Example 2.3.1.

1. For a preorder $\leq$ the relations $(\leq^G)^\sim$ and $\geq^G$ are in general not equal.
2. For an order $\leq$ the relation $\leq^G$ is in general not an order.
3. For an equivalence $=$ the relation $=^G$ is in general not an equivalence.

Proof. The counterexamples are given for the familiar order on $\mathbb{N}$ and the extension $\mathcal{P}c$. We use theorem 2.3.2.2 and some calculus.

1. We have $\{2\} \leq^{\mathcal{P}c} \{1,3\}$ and $\{1,3\} \not\geq^{\mathcal{P}c} \{2\}$. This implies that the relations $(\leq^{\mathcal{P}c})^\sim$ and $\geq^{\mathcal{P}c}$ on $\mathcal{P}.(\mathbb{N})$ are not equal.

2. We have $\{1,2,3\} \leq^{\mathcal{P}c} \{1,3\}$ and $\{1,3\} \leq^{\mathcal{P}c} \{1,2,3\}$. This implies that the relation $\leq^{\mathcal{P}c}$ on $\mathbb{N}$ is not an order.
3. The identity relation $1_{\mathbb{N}}$ is an equivalence, while $1_{\mathbb{N}}^{\mathcal{P}c} \equiv \subseteq$ obviously is not.

(End of example)

This example shows that in combination with relators, we have to be careful with the use of the notation "$\geq$" for the relation $\leq^{\sim}$. For example, for an extension G the relations $(\leq^{\sim})^G$ and $\geq^G$ are equal, but the relations $(\leq^G)^{\sim}$ and $\geq^G$ are not.

We prove that the extensions $\mathcal{O}$, $\mathcal{P}c$, and $\mathcal{P}$ are monotonic, which implies that the corresponding relators are also monotonic (theorem 2.2.2).

Theorem 2.3.3 *The extensions $\mathcal{O}$, $\mathcal{P}c$, and $\mathcal{P}$ are monotonic.*

Proof. Let A, B, and C be sets, $f : A \to C$, and $g : B \to C$.

1. Let $(p, I) \in \mathcal{O}.A$ and $(q, J) \in \mathcal{O}.B$ be such that $\mathcal{O}.f.(p, I) \equiv \mathcal{O}.g.(q, J)$, or equivalently (use definition 1.4.3) $p \equiv q$ and $f \circ I \equiv g \circ J$. We have to prove

$$(p, I)\{(a, b) \in A \times B \mid f.a \equiv g.b\}^{\mathcal{O}}(q, J) .$$

From theorem 2.3.2.1 we see that this is equivalent to

$$(p \equiv q) \wedge I\{(a, b) \in A \times B \mid f.a \equiv g.b\}J .$$

This follows directly from $p \equiv q$ and $f \circ I \equiv g \circ J$.

2. Let $u \in \mathcal{P}.A$ and $v \in \mathcal{P}.B$ (so $\#u < \rho$ and $\#v < \rho$). Then

$$\begin{array}{ll} & \mathcal{P}.f.u \subseteq \mathcal{P}.g.v \\ \Leftrightarrow & \quad \{ \text{ definition 1.4.4.1, calculus } \} \\ & \forall a \in u\, \exists b \in v\, (f.a \equiv g.b) \\ \Leftrightarrow & \quad \{ \text{ theorem 2.3.2.2 } \} \\ & u\{(a, b) \in A \times B \mid f.a \equiv g.b\}^{\mathcal{P}c} v \end{array}$$

3. Let $u \in \mathcal{P}.A$ and $v \in \mathcal{P}.B$. Then

$$\begin{array}{ll} & \mathcal{P}.f.u \equiv \mathcal{P}.g.v \\ \Leftrightarrow & \quad \{ \text{ calculus } \} \\ & \mathcal{P}.f.u \subseteq \mathcal{P}.g.v \wedge \mathcal{P}.g.v \subseteq \mathcal{P}.f.u \\ \Rightarrow & \quad \{ \text{ two times part 2, Put } R \equiv \{(a, b) \mid f.a \equiv g.b\} \text{ } \} \\ & uR^{\mathcal{P}c}v \wedge v(R^{\sim})^{\mathcal{P}c}u \\ \Leftrightarrow & \quad \{ \text{ theorem 2.3.1 } \} \\ & uR^{\mathcal{P}}v \end{array}$$

□

We know that the extensions $\mathcal{Q}$ and $\mathcal{Q}c$ are compositions of $\mathcal{P}$ and $\mathcal{O}$, and $\mathcal{P}c$ and $\mathcal{O}$, respectively. In the above theorem we see that all the components are monotonic, and from this we prove that the extensions $\mathcal{Q}$ and $\mathcal{Q}c$ are monotonic, and give explicit expressions for the corresponding relators.

Corollary 2.3.1 *The extensions $\mathcal{Q}c$ and $\mathcal{Q}$ are monotonic, and for a relation $R \subseteq A \times B$ we have $R^{\mathcal{Q}c} \equiv (R^{\mathcal{O}})^{\mathcal{P}c}$ and $R^{\mathcal{Q}} \equiv R^{\mathcal{Q}c} \cap R^{(\mathcal{Q}c^{\sim})}$. Explicitly: for $u \in \mathcal{Q}.A$ and $v \in \mathcal{Q}.B$ we have*

- $uR^{\mathcal{Q}c}v$ *is equivalent to* $\forall (p, I) \in u\, \exists (q, J) \in v\, (p \equiv q \wedge IRJ)$.
- $uR^{\mathcal{Q}}v$ *is equivalent to*

$$\forall (p, I) \in u\, \exists (q, J) \in v\, (p \equiv q \wedge IRJ) \quad \wedge \quad \forall (q, J) \in v\, \exists (p, I) \in u\, (p \equiv q \wedge IRJ) .$$

Proof. Case $\mathcal{Q}c$ follows from definition 2.3.1.2 and corollary 2.2.1.2 . Case $\mathcal{Q}$ follows from definition 2.3.1.2, theorem 2.3.1, and the easily proved fact $\mathcal{Q}c^{\sim} \equiv \mathcal{P}c^{\sim} \circ \mathcal{O}$. The explicit expressions follow from the explicit expressions for $R^{\mathcal{O}}$ and $\mathcal{R}^{\mathcal{P}c}$ in the above theorem, and theorem 2.3.2.2 . $\square$